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The effect of heat transfer on the stability of laminar boundary layers

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Abstract—Classical linear stability theory is extended to include the effects of temperature dependent viscosity and density. From an asymptotic point of view, i.e. after a Taylor series expansion of all properties with respect to temperature and pressure, they turn out to be the leading order variable property effects for forced convection at low speeds. In an asymptotic approach assuming small heat transfer rates, the two property effects acting on the basic flow and its perturbations are well separated from each other. The asymptotic solutions hold for all Newtonian fluids. Examples are given for air and water. The numerical results are in good agreement with experimental data from the heat transfer literature.

1. INTRODUCTION

In an effort to better understand the nature and origin of turbulence, numerous theoretical and experimental investigations of the stability of laminar boundary layer flows have been made: see Stuart [1], Bayly *et al.* [2], Herbert [3], Huerre and Monkewitz [4] for a review.

But, among all these studies only a few have taken into account the effects of variable properties, even though non-constant properties can have a strong effect on the critical Reynolds number. For a heated flat plate boundary layer in water, for example, Wazzan *et al.* [5] found that the critical Reynolds number varies between 520 and nearly 16 000 (based on the displacement thickness). Thus there is a considerable potential for transition control with technical applications in various fields: see, for example, Morkovin and Reshotko [6] for an overview on transition control for drag reduction.

Since the transition process can be controlled only in its first stage of development, a theoretical approach based on the linear stability analysis is sufficient and adequate. The aim of the present study is to provide results as general as possible, i.e. not restricted to specific fluids or distinct heat transfer rates. The method is that of an asymptotic approach for small heat transfer rates which provides results that hold for all Newtonian fluids. The general theory is outlined in Herwig and Schäfer [7]: in what follows it is applied to forced convection boundary layer flows at low speeds.

Temperature dependence of viscosity μ^* and density ρ^* turn out to be the most important (first order) property effects that have to be taken into account, when small heat transfer rates are assumed. This follows from a Taylor series expansion of all properties with respect to temperature. With a^* representing one of the physical properties ρ^*, μ^*, k^* and c_p^*, the Taylor series expansion reads (dimensional quantities are starred)

$$a = \frac{a^*}{a_R^*} = 1 + \varepsilon K_{aT}\Theta + O(\varepsilon^2) \quad \text{for } \varepsilon \to 0 \qquad (1)$$

with

$$\varepsilon = \frac{\Delta T_R^*}{T_R^*}, \quad K_{aT} = \left[\frac{\partial a^*}{\partial T^*}\frac{T^*}{a^*}\right]_R. \qquad (2)$$

Here ε is introduced as a small (perturbation) parameter. The leading order variable property effects are represented by terms $\varepsilon K_{aT}\Theta$. Since only ρ^* and μ^* appear in the momentum equation, only $\varepsilon K_{\rho T}$- and $\varepsilon K_{\mu T}$-effects must be considered, leaving εK_{kT} and εK_{cT} as higher order effects.

Similar considerations for the pressure dependence of the properties yield that its influence is negligible in the limit $Ma \to 0$ (flows at low speed): for details see Herwig and Schäfer [7].

2. BASIC EQUATIONS

For constant properties, the fundamental differential equation of the linear stability theory is the Orr–Sommerfeld equation (OS equation): see, for example, Schlichting [8]. From this equation the amplitude functions as well as two real eigenvalues of the (small) perturbations can be determined. Perturbations are assumed two-dimensional since they are

NOMENCLATURE

$A_{\rho T}, A_{\mu T}$	auxiliary functions, equation (45)
c_i	amplification rate
c_r	phase velocity
c_p	specific heat at constant pressure
f	flat plate stream function
k	thermal conductivity
$K_{\mu T}$	nondimensional viscosity gradient, equation (13)
$K_{\rho T}$	nondimensional density gradient, equation (13)
Ma	Mach number
p	pressure
Pr	Prandtl number, equation (8)
q_w	wall heat flux
Re	Reynolds number, equation (8)
T	temperature
u	streamwise velocity
$\hat{u}$	amplitude function of the streamwise velocity
U_R	reference velocity
v	velocity normal to the wall
$\hat{v}$	amplitude function of the velocity normal to the wall
x, y	Cartesian coordinates.

Greek symbols

α	wavelength parameter, equation (3)
ε	perturbation parameter, equations (2), (14)
Θ	nondimensional temperature $(T^*-T_R^*)/\Delta T_R^*$
μ	viscosity
$\hat{\mu}$	amplitude function of the viscosity
ρ	density
$\hat{\rho}$	amplitude function of the density
φ	stream function.

Superscripts

$*$	dimensional quantity
$\bar{}$	mean value
$'$	fluctuating quantity
$'$	derivative with respect to y
$\hat{}$	complex quantity.

Subscripts

c	critical
i	imaginary part
L	at $x^* = L^*$
r	real part
R	reference condition
w	wall
0	zero order
$1\rho, 1\mu$	first order.

more 'dangerous' (earlier instability) than three-dimensional ones: for details see the fundamental study of Squire [9] and its extension to variable property flows by Yih [10].

For the present study an extended version of the OS equation is needed which holds for temperature dependent density and viscosity. Due to the temperature dependence of ρ^* and μ^* this modified OS equation must be supplemented by the thermal energy equation for the disturbance.

With the common assumptions (e.g. Schlichting [8]) that all quantities are decomposed into a mean value, $\bar{a}^*$, and a superimposed disturbance a'^* and that any arbitrary two-dimensional disturbance can be expanded in a Fourier series, a single oscillation of wave number α^* is assumed to be of the form (temporal stability)

$$\hat{a}'^*(x^*, y^*, t^*) = \hat{a}^*(y^*) \exp\left[i\alpha^*(x^*-\hat{c}^*t^*)\right]. \quad (3)$$

Here, a^* represents the velocity components u^* and v^* (two-dimensional flow) and the pressure p^*. When variable properties are involved, it also represents these properties, i.e. in our case density ρ^*, viscosity μ^*, as well as the temperature T^*. All complex quantities are marked by the symbol $\hat{}$. In the complex quantity $\hat{c}^*$, with

$$\hat{c}^* = c_r^* + ic_i^* \quad (4)$$

c_r^* denotes the phase velocity, whereas c_i^* determines the degree of amplification or damping.

From the Navier–Stokes equations and the thermal energy equation (both for variable ρ^* and μ^*), together with the continuity equation, the following linearized differential equations for the dimensionless amplitude functions $\hat{u}(y)$, $\hat{v}(y)$ and $\hat{\Theta}(y)$ are derived by inserting equation (3), subtracting the mean flow equations, and eliminating the pressure in the momentum equations. All quantities are nondimensionalized with a reference length L_R^*, a reference velocity U_R^* and a reference temperature difference ΔT_R^* which may be specified later.

$$\hat{\rho}(\bar{u}-\hat{c}) + \bar{\rho}'\frac{\hat{v}}{i\alpha} + \bar{\rho}\left[\hat{u} + \frac{\hat{v}'}{i\alpha}\right] = 0 \quad (5)$$

$$\begin{aligned}\bar{\rho}\left\{(\bar{u}-\hat{c})\left(\hat{u}'+\alpha^2\frac{\hat{v}}{i\alpha}\right)+\bar{u}''\frac{\hat{v}}{i\alpha}+\bar{u}'\left[\hat{u}+\frac{\hat{v}'}{i\alpha}\right]\right\} \\ +\bar{\rho}'\left\{\hat{u}(\bar{u}-\hat{c})+\bar{u}'\frac{\hat{v}}{i\alpha}\right\}+\frac{i}{\alpha\,Re}\left\{\bar{\mu}\left(\hat{u}'''-\alpha^2\hat{u}'\right.\right. \\ \left.+\alpha^2\frac{\hat{v}''}{i\alpha}-\alpha^4\frac{\hat{v}}{i\alpha}\right)+2\bar{\mu}'(\hat{u}''-\alpha^2\hat{u})+\bar{\mu}''\left(\hat{u}'-\alpha^2\frac{\hat{v}}{i\alpha}\right) \\ \left.+\hat{\mu}(\bar{u}'''+\alpha^2\bar{u}')+2\hat{\mu}'\bar{u}''+\hat{\mu}''\bar{u}'\right\} = 0 \quad (6)\end{aligned}$$

Table 1. Comparison of methods: (a) property expansion method and (b) direct solution method

	Advantage	Disadvantage
Method (a): series expansion of $\rho^*(T^*)$, $\mu^*(T^*)$	Results hold for all sets of property laws and all (small) heat transfer rates	Asymptotically correct only for small heat transfer rates
Method (b): specific $\rho^*(T^*)$, $\mu^*(T^*)$	No restrictions with respect to the heat transfer rate	Results hold only for one specific set of property laws and one heat transfer rate

$$\bar{\rho}\left\{(\bar{u}-\hat{c})\hat{\Theta}+\bar{\Theta}'\frac{\hat{v}}{i\alpha}\right\}+\frac{i}{\alpha\, Re\, Pr}(\hat{\Theta}''-\alpha^2\hat{\Theta})=0 \quad (7)$$

with

$$Re=\frac{\rho_R^* U_R^* L_R^*}{\mu_R^*},\quad Pr=\frac{\mu_R^* c_p^*}{k^*} \quad (8)$$

as the Reynolds and Prandtl number, respectively. As usual quadratic terms are neglected (linear stability theory) and mean flow quantities are assumed to be only y-dependent (parallel flow assumption). The notation a' here denotes the derivative of quantity a with respect to y.

The associated boundary conditions are

$$y=0 \quad \hat{u}=\hat{v}=\hat{\Theta}=0 \quad (9)$$

$$y\to\infty \quad \hat{u}=\hat{v}=\hat{\Theta}=0. \quad (10)$$

Equations (5)–(7) are three equations for $\hat{u}$, $\hat{v}$ and $\hat{\Theta}$. They can be solved by two methods:

(a) after an asymptotic expansion of all dependent variables based on the Taylor series expansion of $\rho^*(T^*)$ and $\mu^*(T^*)$ (property expansion method).
(b) directly, i.e. after specific property laws $\rho^*=\rho^*(T^*)$ and $\mu^*=\mu^*(T^*)$ are introduced (direct solution method).

Table 1 lists the advantages and disadvantages of methods (a) and (b). All theoretical studies involving variable properties that we could find in the literature use method (b). Typical examples are studies like that of Wazzan *et al.* [5] and that of Lee *et al.* [11]. For reasons that will be further explained in the discussion of this study, we definitely prefer method (a). Nevertheless, method (b) may always serve as a standard of comparison for expansion method (a).

3. PROPERTY EXPANSION METHOD

The perturbation solution of the stability problem is based on the Taylor series expansions of ρ^* and μ^*. According to equation (1) they read

$$\rho=1+\varepsilon K_{\rho T}\Theta+O(\varepsilon^2) \quad (11)$$

$$\mu=1+\varepsilon K_{\mu T}\Theta+O(\varepsilon^2) \quad (12)$$

with

$$K_{\rho T}=\left[\frac{\partial\rho^*}{\partial T^*}\frac{T^*}{\rho^*}\right]_R \quad K_{\mu T}=\left[\frac{\partial\mu^*}{\partial T^*}\frac{T^*}{\mu^*}\right]_R \quad (13)$$

and the perturbation parameter

$$\varepsilon=\frac{(T_w^*-T_\infty^*)_{L^*}}{T_\infty^*} \quad (14)$$

which here is specified for a case with constant wall heat flux q_w^*. When only linear terms of the expansions (11) and (12) are taken into account, we will call this a linear perturbation theory. What follows is a linear analysis, but extension to higher orders with respect to ε is straightforward.

The K_{aT}-parameters are properties of the fluid. Table 2 gives specific numbers for air and water under standard conditions.

Due to the decomposition $a=\bar{a}+\hat{a}\exp[i\alpha(x-\hat{c}t)]$ the property expansions read

$$\bar{\rho}=1+\varepsilon K_{\rho T}\bar{\Theta}+O(\varepsilon^2) \quad \hat{\rho}=\varepsilon K_{\rho T}\hat{\Theta}+O(\varepsilon^2) \quad (15)$$

$$\bar{\mu}=1+\varepsilon K_{\mu T}\bar{\Theta}+O(\varepsilon^2) \quad \hat{\mu}=\varepsilon K_{\mu T}\hat{\Theta}+O(\varepsilon^2). \quad (16)$$

The mean flow field is affected by variable property effects through $\bar{\rho}$ and $\bar{\mu}$, whereas the stability equations (5)–(7) are affected by the mean as well as by the disturbance parts of the properties.

Equations (11) and (12) suggest an expansion of all mean flow and disturbance quantities of the general form

$$a=a_0+\varepsilon(K_{\rho T}a_{1\rho}+K_{\mu T}a_{1\mu})+O(\varepsilon^2) \quad (17)$$

with the quantity a representing: $\bar{u}$, $\hat{u}$, $\bar{v}$, $\hat{v}$, $\bar{p}$, $\hat{p}$, $\bar{\Theta}$, $\hat{\Theta}$,

Table 2. Prandtl number and K_{aT}-values of air and water for $T_R^*=293$ K, $p_R^*=1$ bar

	Pr	$K_{\rho T}$	$K_{\mu T}$
Air	0.717	−1.000	0.775
Water	7.010	0.057	−7.132

$\hat{c}$. According to equation (17) the complex parameter $\hat{c}$ is

$$\hat{c} = \hat{c}_0 + \varepsilon(K_{\rho T}\hat{c}_{1\rho} + K_{\mu T}\hat{c}_{1\mu}) + O(\varepsilon^2) \tag{18}$$

with an imaginary part (amplification rate)

$$c_i = c_{0i} + \varepsilon(K_{\rho T}c_{1\rho i} + K_{\mu T}c_{1\mu i}) + O(\varepsilon^2). \tag{19}$$

The expansion of the parameter $\hat{c}$ in the same way as all other functions is a crucial step in the present theory. Determination of $\hat{c}_{1\rho}$ and $\hat{c}_{1\mu}$ from the first order equations below is the final objective of our theory.

Inserting the expansions (15)–(17) into the equations for the mean flow as well as in those for the perturbations, equations (5)–(7), and collecting terms with respect to $\varepsilon K_{\rho T}$ and $\varepsilon K_{\mu T}$, gives the asymptotic equations of the stability problem.

4. APPLICATION OF BOUNDARY LAYERS

The property expansion method now is applied to boundary layer flows. A numerical example will be given for the flat plate boundary layer with constant wall heat flux q_w^*. For other boundary layers (with pressure gradient) only the mean flow quantities must be changed. No additional terms appear in the equations for the amplitude functions of the perturbation. Therefore the flat plate case is an adequate example.

4.1. *Mean flow quantities*

In the boundary layer equations for the mean flow the expansions for $\bar{\rho}$ and $\bar{\mu}$ according to equations (15) and (16) are inserted together with the expansions for all dependent quantities, cf. equation (17).

For the example of the flat plate the equations are, cf. Gersten and Herwig [12], with the self-similar stream function $f(y)$, $y = y^*/L_R^*$, $L_R^* = [\mu_R^* x^*/(\rho_R^* U_\infty^*)]^{1/2}$, $\bar{\rho}\bar{u} = f' = \mathrm{d}f/\mathrm{d}y$ and $\bar{\Theta} = (T^*(x^*) - T_\infty^*)/(\bar{T}_w^*(x^*) - T_\infty^*)$:

$$\left[\bar{\mu}\left(\frac{f'}{\bar{\rho}}\right)'\right]' + \tfrac{1}{2}f\left(\frac{f'}{\bar{\rho}}\right)' = 0 \tag{20}$$

$$\bar{\Theta}'' + \tfrac{1}{2}Pr\,(f\bar{\Theta}' - f'\bar{\Theta}) = 0 \tag{21}$$

with the boundary conditions:

$$y = 0 \quad f = f' = \bar{\Theta} - 1 = 0 \quad y \to \infty \quad f' - 1 = \bar{\Theta} = 0. \tag{22}$$

Inserting

$$\bar{\mu} = 1 + \varepsilon K_{\mu T}\bar{\Theta} + O(\varepsilon^2) \quad \bar{\rho} = 1 + \varepsilon K_{\rho T}\bar{\Theta} + O(\varepsilon^2)$$

$$f = f_0 + \varepsilon(K_{\rho T}f_{1\rho} + K_{\mu T}f_{1\mu}) + O(\varepsilon^2) \quad \bar{\Theta} = \bar{\Theta}_0 + O(\varepsilon)$$

into equations (20) and (21) gives

zero order:

$$f_0''' + \tfrac{1}{2}f_0 f_0'' = 0 \tag{23}$$

$$\bar{\Theta}_0'' + \tfrac{1}{2}Pr(f_0\bar{\Theta}_0' - f_0'\bar{\Theta}_0) = 0 \tag{24}$$

1st order:

$$f_{1\rho}''' + \tfrac{1}{2}(f_{1\rho}''f_0 + f_{1\rho}f_0'') = (\bar{\Theta}_0'f_0)' + \bar{\Theta}_0'(f_0'' + \tfrac{1}{2}f_0f_0') \tag{25}$$

$$f_{1\mu}''' + \tfrac{1}{2}(f_{1\mu}''f_0 + f_{1\mu}f_0'') = -(\bar{\Theta}_0 f_0'')' \tag{26}$$

with all first order boundary conditions equal to zero. From the solution for f the velocity components are $\bar{u}_0 = f_0'$, $\bar{u}_{1\rho} = f_{1\rho}' - \bar{u}_0\bar{\Theta}_0$ and $\bar{u}_{1\mu} = f_{1\mu}'$.

4.2. *Amplitude functions*

The stability equations (5)–(7), from which the amplitude functions $\hat{u}$, $\hat{v}$ and $\hat{\Theta}$ can be determined, are now subject to a perturbation procedure similar to that of the mean flow. The above-mentioned amplitude functions are expanded according to (17). The amplitude functions $\hat{\rho}$ and $\hat{\mu}$, cf. equations (15) and (16), are (due to their temperature dependence)

$$\hat{\rho} = \varepsilon K_{\rho T}\hat{\Theta}_0 + O(\varepsilon^2) \tag{27}$$

$$\hat{\mu} = \varepsilon K_{\mu T}\hat{\Theta}_0 + O(\varepsilon^2). \tag{28}$$

Inserting all expansions and collecting terms of equal magnitude with respect to $\varepsilon K_{\rho T}$ and $\varepsilon K_{\mu T}$ leads to the following set of asymptotic stability equations. For convenience stream functions are introduced for the zero order equation and the viscosity first order equation, defined as

$$\hat{\varphi}_0 = -\frac{\hat{v}_0}{i\alpha} \quad \hat{\varphi}_0' = \hat{u}_0 \tag{29}$$

$$\hat{\varphi}_{1\mu} = -\frac{\hat{v}_{1\mu}}{i\alpha} \quad \hat{\varphi}_{1\mu}' = \hat{u}_{1\mu}. \tag{30}$$

The equations are

zero order:

$$(\bar{u}_0 - \hat{c}_0)(\hat{\varphi}_0'' - \alpha^2\hat{\varphi}_0) - \bar{u}_0''\hat{\varphi}_0 + \frac{i}{\alpha Re}(\hat{\varphi}_0'''' - 2\alpha^2\hat{\varphi}_0'' + \alpha^4\hat{\varphi}_0) = 0 \tag{31}$$

$$(\bar{u}_0 - \hat{c}_0)\hat{\Theta}_0 + \frac{i}{\alpha Re\,Pr}(\hat{\Theta}_0'' - \alpha^2\hat{\Theta}_0) = \bar{\Theta}_0'\hat{\varphi}_0. \tag{32}$$

1st order (density):

$$\hat{u}_{1\rho} + \frac{\hat{v}_{1\rho}'}{i\alpha} + \hat{\Theta}_0(\bar{u}_0 - \hat{c}_0) + \bar{\Theta}_0'\frac{\hat{v}_0}{i\alpha} = 0 \tag{33}$$

$$(\bar{u}_0 - \hat{c}_0)\left[\hat{u}_{1\rho}' + \alpha^2\frac{\hat{v}_{1\rho}}{i\alpha}\right] + \bar{u}_0''\frac{\hat{v}_{1\rho}}{i\alpha} + \frac{i}{\alpha Re}\left[\hat{u}_{1\rho}''' - 2\alpha^2\hat{u}_{1\rho}' - \alpha^4\frac{\hat{v}_{1\rho}}{i\alpha}\right] = -(\bar{u}_{1\rho} - \hat{c}_{1\rho})\left[\hat{u}_0' + \alpha^2\frac{\hat{v}_0}{i\alpha}\right] - \bar{u}_{1\rho}''\frac{\hat{v}_0}{i\alpha}$$

$$+\frac{i}{\alpha\, Re}\left[\alpha^2\hat{\Theta}_0'(\bar{u}_0-\hat{c}_0)+\alpha^2\hat{\Theta}_0\hat{u}_0' +\alpha^2\bar{\Theta}_0''\frac{\hat{v}_0}{i\alpha}+\alpha^2\bar{\Theta}_0'\frac{\hat{v}_0'}{i\alpha}\right]$$

$$-\bar{\Theta}_0\bar{u}_0''\frac{\hat{v}_0}{i\alpha}-\bar{\Theta}_0\left[\hat{u}_0'+\alpha^2\frac{\hat{v}_0}{i\alpha}\right](\bar{u}_0-\hat{c}_0)$$

$$+(\bar{u}_0-\hat{c}_0)[\bar{u}_0'\hat{\Theta}_0-\bar{\Theta}_0'\hat{u}_0]. \quad (34)$$

1st order (viscosity):

$$(\bar{u}_0-\hat{c}_0)(\hat{\varphi}_{1\mu}''-\alpha^2\hat{\varphi}_{1\mu})-\bar{u}_0''\hat{\varphi}_{1\mu} +\frac{i}{\alpha\, Re}(\hat{\varphi}_{1\mu}''''-2\alpha^2\hat{\varphi}_{1\mu}''+\alpha^4\hat{\varphi}_{1\mu})$$

$$= -(\bar{u}_{1\mu}-\hat{c}_{1\mu})(\hat{\varphi}_0''-\alpha^2\hat{\varphi}_0)+\bar{u}_{1\mu}''\hat{\varphi}_0$$

$$-\frac{i}{\alpha\, Re}[\bar{\Theta}_0(\hat{\varphi}_0''''-2\alpha^2\hat{\varphi}_0''+\alpha^4\hat{\varphi}_0)$$

$$+2\bar{\Theta}_0'(\hat{\varphi}_0'''-\alpha^2\hat{\varphi}_0')+\bar{\Theta}_0''(\hat{\varphi}_0''+\alpha^2\hat{\varphi}_0)$$

$$+\hat{\Theta}_0(\bar{u}_0'''+\alpha^2\bar{u}_0')+2\hat{\Theta}_0'\bar{u}_0''+\hat{\Theta}_0''\bar{u}_0'] \quad (35)$$

with the associated boundary conditions

$$y=0:\quad \hat{\varphi}_0=\hat{\varphi}_0'=\hat{\Theta}_0'=\hat{u}_{1\rho}=\hat{v}_{1\rho}=\hat{\varphi}_{1\mu}=\hat{\varphi}_{1\mu}'=0 \quad (36)$$

$$y\to\infty:\quad \hat{\varphi}_0=\hat{\varphi}_0'=\hat{\Theta}_0=\hat{u}_{1\rho}=\hat{v}_{1\rho}=\hat{\varphi}_{1\mu}=\hat{\varphi}_{1\mu}'=0. \quad (37)$$

Equation (31) is the classical OS equation for constant properties which is the zero order equation of the asymptotic expansion with respect to ε. The solution procedure starts with this equation. The first order equations for density and viscosity are affected by its solution but are, due to the expansions, independent of each other.

5. NUMERICAL SOLUTIONS

The mathematical nature of the asymptotic stability equations is distinctly different for each of the equations (31)–(35).

The classical OS equation is an eigenvalue problem which owing to its stiffness is difficult to solve numerically [13]. One way to deal with this stiffness is the so-called Gram–Schmidt orthonormalization: for details see Mack [14] and Herwig and Schäfer [7]. One feature of this solution procedure is that the amplitude functions are determined only as piecewise steady functions in subregions of orthonormalization. Since a continuous function $\hat{\varphi}_0$ is needed in the zero order energy equation as well as in the first order momentum equations, it must be restored in a way described in Herwig and Schäfer [7]. Figure 1 shows the real and imaginary parts of $\hat{\varphi}_0$ as well as both parts of one of the four fundamental solutions $\hat{\varphi}_{01}$–$\hat{\varphi}_{04}$ from which $\hat{\varphi}_0$ is constructed. The solution parameters prescribed in Fig. 1 were $Re = 301.7$ and $c_{0i} = 0.0$, the two eigenvalues of the solution were $\alpha = 0.1763$ and $c_{0r} = 0.3965$. Actually this case corresponds to our critical Reynolds number (lowest Reynolds number on the neutral curve $c_{0i} = 0.0$) which agrees very well with that of other studies (ref. [8]: $Re_c = 302$; ref. [15]: $Re_c = 303$).

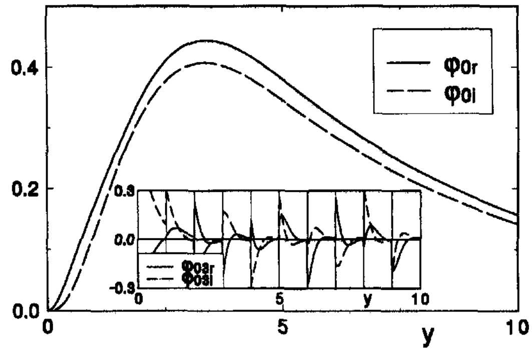

Fig. 1. Zero order amplitude function $\hat{\varphi}_0$; $\hat{\varphi}_0'(0) = 1$ for normalization: $Re = 301.7$, $c_{0i} = 0.0$ ($\alpha = 0.3034$, $c_{0r} = 0.3965$). Inset: Ten subregions of orthonormalization with the two functions φ_{03r} and φ_{03i}. Increasing the number of subregions will increase the computational accuracy.

The thermal energy equation (32) for $\hat{\Theta}_0(y)$ is a nonhomogeneous linear second order differential equation with homogeneous boundary conditions. It is also a stiff differential equation like (31), and was solved by the so-called multiple shooting method [16]. In this method, the whole solution domain is cast into subregions. Then a first step integration is performed starting from assumed boundary conditions in each subregion (taking into account the boundary conditions at the wall and for $y \to \infty$). In subsequent steps, the discontinuities at the boundaries of the subregions are removed so that a continuous function $\hat{\Theta}_0$ results.

The only solution parameter of the energy equation is the Prandtl number. The two eigenvalues of the problem enter the energy equation indirectly through $\hat{\varphi}_0$ which must be known from the OS equation.

In Fig. 2 the real and imaginary parts of the function $\hat{\Theta}_0'$ are shown for $Pr = 0.7$ with $\hat{\varphi}_0$ from Fig. 1 ($Re = 301.7$, $c_{0i} = 0.0$).

The first order equations (33)–(35) are nonhomogeneous differential equations of the form

$$L[\hat{u}_{1\rho},\hat{v}_{1\rho},\hat{c}_0] = f(\hat{u}_0,\hat{v}_0,\hat{\Theta}_0,\hat{c}_1) \quad (38)$$

$$L[\hat{\varphi}_{1\mu},\hat{c}_0] = f(\hat{\varphi}_0,\hat{\Theta}_0,\hat{c}_1) \quad (39)$$

with the OS differential operator L given by one of the two equivalent forms

$$L[\hat{u},\hat{v},\hat{c}_0] \quad \text{or} \quad L[\hat{\varphi},\hat{c}_0]$$

with

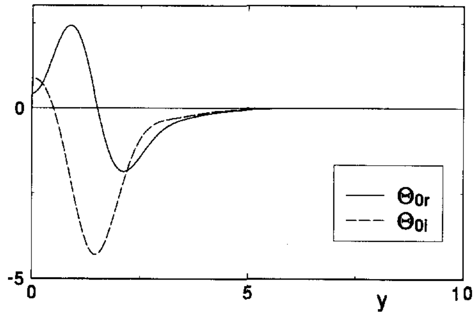

Fig. 2. Zero order amplitude function $\hat{\Theta}_0$: $Pr = 0.7$ ($Re = 301.7$, $c_{0i} = 0.0$).

$$L[\hat{\varphi}, \hat{c}_0] = (\bar{u}_0 - \hat{c}_0)(\hat{\varphi}'' - \alpha^2 \hat{\varphi}) - \bar{u}_0'' \hat{\varphi} + \frac{i}{\alpha\, Re}(\hat{\varphi}'''' - 2\alpha^2 \hat{\varphi}'' + \alpha^4 \hat{\varphi}) \quad (40)$$

for example. Specific values of $\hat{c}_{1\rho}$ and $\hat{c}_{1\mu}$ must be found for which equations (38) and (39) have solutions. These complex constants are solution parameters, equivalent to the eigenvalues of the zero order (homogeneous) problem. The corresponding solutions of the first order equations will be denoted by $\hat{u}_{1\rho p}$, $\hat{v}_{1\rho p}$ and $\hat{\varphi}_{1\mu p}$, respectively (p for particular solution). Their general solutions are

$$\hat{u}_{1\rho} = \hat{u}_{1\rho p} + \hat{C}_\rho \hat{u}_0 \quad \hat{v}_{1\rho} = \hat{v}_{1\rho p} + \hat{C}_\rho \hat{v}_0 \quad (41)$$

$$\hat{\varphi}_{1\mu} = \hat{\varphi}_{1\mu p} + \hat{C}_\mu \hat{\varphi}_0 \quad (42)$$

since the zero order solution $\hat{u}_0$, $\hat{v}_0$ (or $\hat{\varphi}_0$) satisfies $L[\hat{u}, \hat{v}, \hat{c}_0] = 0$ (or $L[\hat{\varphi}, \hat{c}_0] = 0$). Due to the undetermined constants $\hat{C}_\rho$ and $\hat{C}_\mu$ in equations (41) and (42) integration can start from the wall, fixing $\hat{u}'_{1\rho}(0)$ and $\hat{\varphi}''_{1\mu}(0)$ arbitrarily, for example. Integration was again performed by the multiple shooting method, using $\hat{c}_{1\rho}$ and $\hat{c}_{1\mu}$ as shooting parameters in the respective equations.

The process of determining the first order parameters $\hat{c}_{1\rho}$ and $\hat{c}_{1\mu}$ is linear, since $\hat{u}_{1\rho}$, $\hat{v}_{1\rho}$ and $\hat{\varphi}_{1\mu}$ as well as their derivatives are not multiplied by $\hat{c}_{1\rho}$ and $\hat{c}_{1\mu}$, respectively. Therefore, no iteration is needed to determine $\hat{c}_{1\rho}$ and $\hat{c}_{1\mu}$. This is different from the process of determining $\hat{c}_0$, since φ_0 is multiplied by $\hat{c}_0$ in equation (31). In Fig. 3 the real and imaginary parts of the amplitude functions $\hat{u}_{1\rho}$, $\hat{v}_{1\rho}$ and $\hat{\varphi}_{1\mu}$ are shown for the same parameters ($Pr = 0.7$; $Re = 301.7$; $c_i = 0.0$) as in the previous figures. For normalization we have set $\hat{u}'_{1\rho}(0) = \hat{\varphi}''_{1\mu}(0) = (1 + i1)$. The solution parameters for this case are $\hat{c}_{1\rho} = -0.143 - i0.062$ and $\hat{c}_{1\mu} = 0.486 + i0.072$.

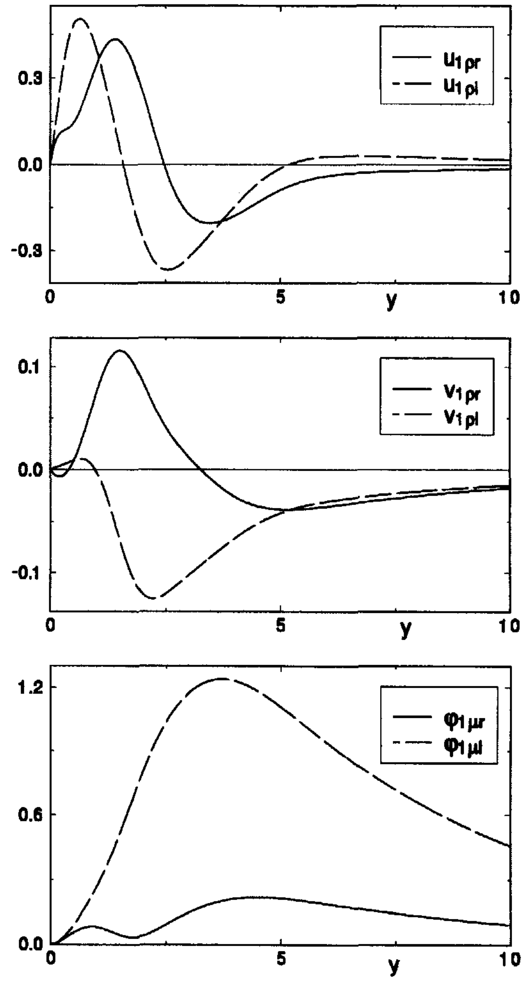

Fig. 3. First order amplitude functions for $Pr = 0.7$; $Re = 301.7$; $c_{0i} = 0.0$ for (a) $\hat{u}_{1\rho}$, (b) $\hat{v}_{1\rho}$ and (c) $\hat{\varphi}_{1\mu}$.

6. DETERMINATION OF Re_c

The critical Reynolds number Re_c defined as the lowest Reynolds number on the neutral curve ($c_i = 0.0$ in an α–Re diagram) physically describes the onset of the transition process. Below Re_c the flow is stable with respect to oscillations of all wave numbers. The effect of heat transfer on the stability of laminar boundary layers therefore can be characterized by the way it affects the critical Reynolds number. The final objective of our study is to deduce an asymptotic expression for Re_c of the form

$$Re_c = Re_{c0}[1 + \varepsilon(K_{\rho T} A_{\rho T} + K_{\mu T} A_{\mu T}) + O(\varepsilon^2)] \quad (43)$$

with Re_{c0} as the critical Reynolds number without heat transfer ($Re_{c0} = 301.7$ for the flat plate). A particular formula (43) will hold for a specific flow and heat transfer type only (for example: flat plate flow with $q_w = \text{const.}$) but due to the expansion procedure it will hold for:

all (small) heat transfer rates by specifying ε according to the rate of heat transfer under consideration and

all (Newtonian) fluids by specifying $K_{\rho T}$ and $K_{\mu T}$ according to the fluid under consideration (cf. Table 2).

It takes two steps to determine $A_{\rho T}$ and $A_{\mu T}$ from the asymptotic stability equations.

Step (1): Finding the functional dependence

$Re_c = Re_c(\varepsilon K_{\rho T}, \varepsilon K_{\mu T})$ and especially $Re_c(\varepsilon K_{\rho T}, 0)$

$$\text{and } Re_c(0, \varepsilon K_{\mu T}) \quad (44)$$

Step (2) : Determination of

$$A_{\rho T} = \left[\frac{\partial\, Re_c(0,0)}{\partial \varepsilon K_{\rho T}}\right]_{\varepsilon K_{\mu T}} Re_{c0}^{-1}$$

$$\text{and } A_{\mu T} = \left[\frac{\partial\, Re_c(0,0)}{\partial \varepsilon K_{\mu T}}\right]_{\varepsilon K_{\rho T}} Re_{c0}^{-1}. \quad (45)$$

Since Re_c is Prandtl number dependent for $\varepsilon \neq 0$ the coefficients $A_{\rho T}$ and $A_{\mu T}$ will also depend on Pr. This can be taken into account by a proper variation of the Prandtl number in the process of determining the two coefficients.

Step (1), which actually is the major one, will be performed as follows.

Since the critical Reynolds number is reached when $c_i = 0.0$ for just one wave number α (then the line Re = const. touches the neutral curve at the level $\alpha = \alpha_c$ in the α–Re diagram) the condition for Re_c is

$$c_i = c_{0i} + \varepsilon(K_{\rho T} c_{1\rho i} + K_{\mu T} c_{1\mu i}) = 0 \text{ for just one } \alpha. \quad (46)$$

This can be evaluated when c_{0i}, $c_{1\rho i}$ and $c_{1\mu i}$ are known functions of α.

In Fig. 4 they are shown for the flat plate at $Pr = 0.7$ for three different Reynolds numbers. The curves in Fig. 4 are smooth curves through sufficiently dense data points. Each data point itself is the result of a numerical solution of equations (31) for c_{0i}, (33) and (34) for $c_{1\rho i}$ or (35) for $c_{1\mu i}$.

Each Reynolds number case in Fig. 4 will contribute one point to the curves $Re_c(\varepsilon K_{\rho T}, 0)$ and $Re_c(0, \varepsilon K_{\mu T})$ by choosing $\varepsilon K_{\rho T}$ and $\varepsilon K_{\mu T}$ such that $c_{0i} + \varepsilon K_{\rho T} c_{1\rho i} = 0$ and $c_{0i} + \varepsilon K_{\mu T} c_{1\mu i} = 0$, respectively, for just one α. Figure 5 depicts both curves which emerge from a large number of diagrams like Fig. 4.

In step (2) of the procedure one simply has to find the tangents to the two curves in Fig. 5 to determine $A_{\rho T}$ and $A_{\mu T}$ according to equation (45).

After repeating this procedure for different Prandtl numbers we finally get the $A_{\rho T}(Pr)$ and $A_{\mu T}(Pr)$ curves. In Fig. 6 they are shown for the Prandtl number range $0.1 \leqslant Pr \leqslant 10$.

7. DISCUSSION

The final result of the property expansion method, equation (43), clearly reveals how the boundary layer stability is affected by heat transfer across the wall.

For the example of flat plate flow (q_w = const.) we find:

(1) an opposite behaviour for heating ($\varepsilon > 0$) and cooling ($\varepsilon < 0$) of the fluid and

(2) two different variable property effects (density and viscosity) which enhance each other for fluids with opposite signs of $K_{\rho T}$ and $K_{\mu T}$, like air and water: cf. Table 2. In Table 3 this is illustrated by the signs of the various terms in equation (43). From these considerations it follows that boundary layer stabilization ($Re_c > Re_{c0}$) occurs for air by cooling and for water by heating. The amount of stabilization is given by equation (43).

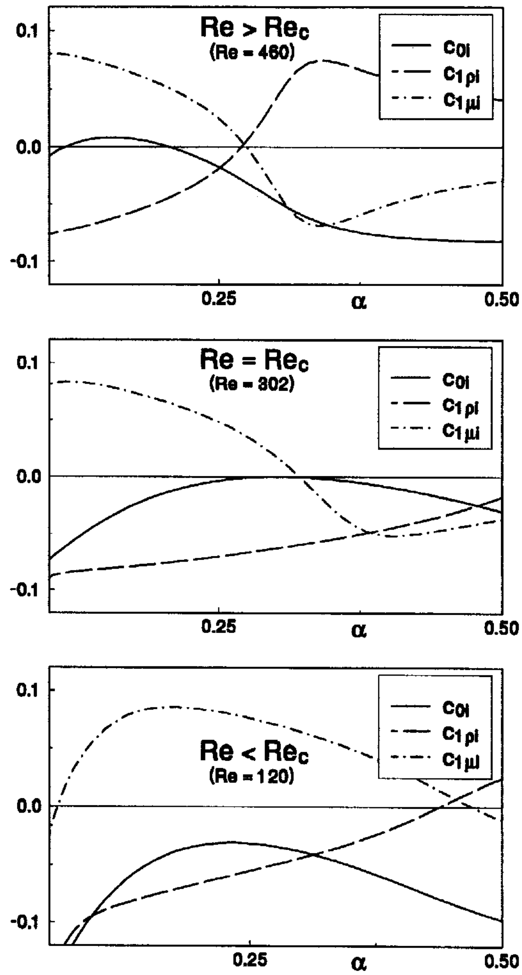

Fig. 4. Eigenvalues c_{0i} and solution parameters $c_{1\rho i}$ and $c_{1\mu i}$ for three different Reynolds numbers: $Pr = 0.7$.

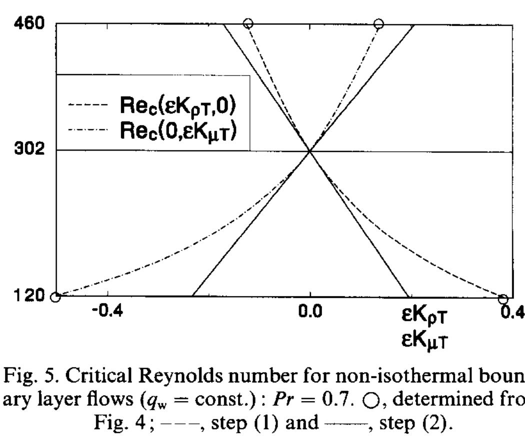

Fig. 5. Critical Reynolds number for non-isothermal boundary layer flows (q_w = const.): $Pr = 0.7$. ○, determined from Fig. 4; ---, step (1) and ——, step (2).

Table 3. Signs of various terms in equation (43) for air and water: flat plate flow (q_w = const.)

	$K_{\rho T}$	$K_{\mu T}$	$K_{\rho T}A_{\rho T}$	$K_{\mu T}A_{\mu T}$	$\varepsilon(K_{\rho T}A_{\rho T}+K_{\mu T}A_{\mu T})$ $\varepsilon>0$ Heating	$\varepsilon<0$ Cooling
Air	−	+	−	−	−	+
Water	+	−	+	+	+	−

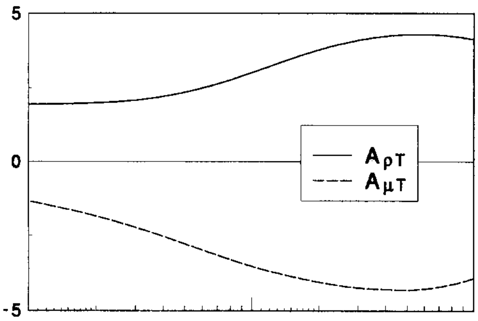

Fig. 6. Critical Reynolds number coefficients for a flat plate boundary layer with q_w = const.

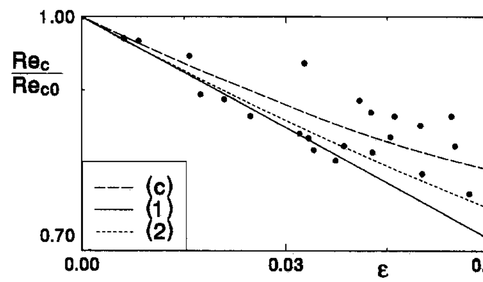

Fig. 8. Comparison with experimental data, Harrison et al. [17]: ●, experimental data; (c), curve fit to the data; (1) property expansion method and (2) direct solution method.

The results of the property expansion method can be compared to those of the direct solution method [no asymptotic expansions, direct solution of equations (5)–(7)] when specific property laws $\rho^*(T^*)$ and $\mu^*(T^*)$ are assumed. As an example we have chosen

$$\rho^* = \rho^*_\infty\left(2-\frac{T^*}{T^*_\infty}\right) \quad \mu^* = \mu^*_\infty\left(0.225+0.775\frac{T^*}{T^*_\infty}\right) \tag{47}$$

which is a good approximation for air. In terms of the property expansion method this case corresponds to $K_{\rho T} = -1.0$ and $K_{\mu T} = 0.775$. Figure 7 shows that both methods asymptotically merge for $T^*_w \to T^*_\infty$ (i.e. $\varepsilon \to 0$). Deviations for increasing temperature differences are due to neglecting $O(\varepsilon^2)$-terms in the property expansion method. They are nonzero even though the property laws are linear [that means K_{aT}-values for all $O(\varepsilon^2)$-terms in equations (11) and (12) are zero, but still terms like $(K_{\rho T}\varepsilon)^2$ appear in the property expansion method]. Curve 1a in Fig. 7 is that of the intermediate result for Re_c [step (1)] in the previous chapter. Though one might argue that it contains more information than the mere tangents [step (2)], Fig. 7 shows that it is not always closer to the exact solution than the direct solution method.

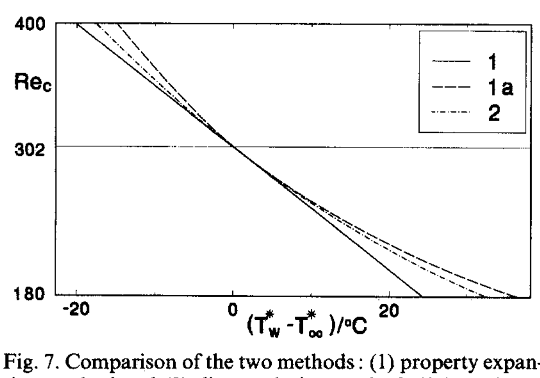

Fig. 7. Comparison of the two methods: (1) property expansion method and (2) direct solution method. (1a) $Re_c(\varepsilon K_{\rho T}, \varepsilon K_{\mu T})$-curve according to equation (44) of the property expansion method.

Finally, we want to compare our results with experimental data. Harrison et al. [17] have measured the influence of wall heating (q_w = const.) on the stability of flat plate boundary layer flow. In Fig. 8 their experimental data together with a proposed curve fit to these data are shown. The results of the property expansion method compare well with these results. Those of the direct solution method are only slightly better. These direct solution results had to be computed by solving equations (5)–(7) for this special case, whereas the property expansion results emerged from a simple application of equation (43) together with Fig. 6 to this special case.

This may emphasize the strength of the property expansion method. There is no need to specify a particular fluid nor a specific heat transfer rate, since the final results in their general form hold for all (Newtonian) fluids and all (small) heat transfer rates.

REFERENCES

1. J. T. Stuart, Instability of flows and their transition to turbulence, *ZFW* **10**, 379–392.
2. B. J. Bayly, S. A. Orszag and H. Herbert, Instability mechanisms in shear-flow transition, *Ann. Rev. Fluid Mech.* **20**, 359–391 (1988).
3. H. Herbert, Secondary instability of boundary layers, *Ann. Rev. Fluid Mech.* **20**, 487–526 (1988).
4. H. Huerre and P. A. Monkewitz, Local and global instabilities in spacially developing flows, *Ann. Rev. Fluid Mech.* **22**, 473–537 (1990).
5. A. R. Wazzan, G. Keltner, T. T. Okamura and A. M.

O. Smith, Spatial stability of stagnation water boundary layer with heat transfer. *Phys. Fluids* **15,** 2114–2118 (1972).
6. M. V. Morkovin and E. Reshotko, Dialogue on progress and issues in stability and transition research. In *Laminar–Turbulent Transition* (Edited by D. Arnal and R. Michel). Springer, Berlin (1990).
7. H. Herwig and P. Schäfer, Influence of variable properties on the stability of two-dimensional boundary layers, *J. Fluid Mech.* **243,** 1–14 (1992).
8. H. Schlichting, *Grenzschicht—Theorie*. G. Braun, Karlsruhe (1982).
9. H. B. Squire, On the stability for three-dimensional disturbances of viscous fluid flow between parallel walls, *Proc. R. Soc. Lond. A* **142,** 621–628 (1933).
10. C.-S. Yih, Stability of two-dimensional parallel flows for three-dimensional disturbances, *Q. Appl. Math.* **12,** 434–435 (1954).
11. H. R. Lee, T. S. Chen and B. F. Armaly, Non-parallel thermal instability of forced convection flow over a heated, non-isothermal horizontal flat plate, *Int. J. Heat Mass Transfer* **33,** 2019–2028 (1990).
12. K. Gersten and H. Herwig, Impuls- und Wärmeübertragung bei variablen Stoffwerten für die laminare Plattenströmung. *Wärme- und Stoffübertragung* **18,** 25–35 (1984).
13. L. M. Mack, Boundary-layer linear stability theory, AGARD Report No. 709 (1984).
14. L. M. Mack, A numerical study of the temporal eigenvalue spectrum of the Blasius boundary layer, *J. Fluid Mech.* **73,** 497–520 (1976).
15. H. Kümmerer, Numerische Untersuchungen zur Stabilität ebener laminarer Grenzschichtströmungen, Dissertation, Universität Stuttgart (1973).
16. J. Stoer and R. Bulirsch, *Einführung in die Numerische Mathematik*, Vol. II. Springer, Berlin (1978).
17. S. B. Harrison, D. J. Mee and T. V. Jones, Experiment on the influence of heating on boundary layer transition in favourable pressure gradients, Proceedings Eurotherm Seminar No. 25, Pau, France (1991).